Padé-Improved Estimate of Perturbative Contributions to Inclusive Semileptonic $b \to u$ Decays

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Padé-approximant methods are used to estimate the three-loop perturbative contributions to the inclusive semileptonic $b \to u$ decay rate. These improved estimates of the decay rate reduce the theoretical uncertainty in the extraction of the CKM matrix element $|V_{ub}|$ from the measured inclusive semileptonic branching ratio.

In this paper we briefly review the development of Padé approximation techniques to QCD quantities satisfying a renormalization group equation,[1] and the application of these techniques to the estimate of three-loop contributions to the inclusive semileptonic $b \to u$ decay rate.[2]

The QCD perturbative contributions to the inclusive semileptonic decay rate $\Gamma\left(b \to X_u \ell^- \bar{\nu}_\ell\right)$ are known to two-loop order.[3] The theoretical prediction of the decay rate is an interesting phenomenological quantity since it depends on the CKM matrix element $|V_{ub}|$. Moreover, the two-loop calculation is mainly sensitive to m_b since the b mass is much larger than final state particle masses (m_u, m_ℓ), and m_c only enters the partial rate $b \to u\ell\bar{\nu}_\ell c\bar{c}$ or in virtual corrections. The $\overline{MS}$ scheme obviates the poor convergence of the perturbative series in on-shell schemes, leading to the following two-loop result for the decay rate for five active flavours:[3]

$$\Gamma\left(b \to X_u \ell^- \overline{\nu}_\ell\right) = K m_b^5(\mu) S\left[x(\mu), L(\mu)\right] \tag{1}$$

$$K \equiv G_F^2 |V_{ub}|^2/192\pi^3 \ , \ x(\mu) = \frac{\alpha(\mu)}{\pi} \ , \ L(\mu) = \log(w) = \log\left[\frac{m_b^2(\mu)}{\mu^2}\right] \tag{2}$$

$$S[x, L] = 1 + x\left(a_0 - a_1 L\right) + x^2\left(b_0 - b_1 L + b_2 L^2\right) \tag{3}$$

$$a_0 = 4.25360 \ , \ a_1 = 5 \ , \ b_0 = 26.7848 \ , \ b_1 = 36.9902 \ , \ b_2 = 17.2917 \tag{4}$$

where μ represents the renormalization scale.

Three-loop corrections to (1) are potentially significant, since at $\mu = m_b = m_b\left(m_b\right) \approx 4.2\,\mathrm{GeV}$ we find

$$\Gamma = K m_b^5\left[1 + 0.30 + 0.14\right] \quad . \tag{5}$$

The general form of $S\left(x, L\right)$ which determines the three-loop order decay rate is

$$S[x,L] = 1 + x\left(a_0 - a_1 L\right) + x^2\left(b_0 - b_1 L + b_2 L^2\right) + x^3\left(c_0 - c_1 L + c_2 L^2 - c_3 L^3\right) \tag{6}$$

The decay rate Γ satisfies a renormalization-group (RG) equation which determines the three-loop coefficients $\{c_1,\ c_2,\ c_3\}$, but leaves the crucial c_0 coefficient undetermined.

$$0 = \mu\frac{d\Gamma}{d\mu} = \left[\mu\frac{\partial}{\partial\mu} + \gamma(x) m_b \frac{\partial}{\partial m_b} + \beta(x)\frac{\partial}{\partial x}\right]\Gamma \tag{7}$$

$$c_1 = 249.592\ ,\ c_2 = 178.755\ ,\ c_3 = 50.9144 \tag{8}$$

Padé approximation methods can be used to estimate the c_i. A comparison of these estimates with the RG-determined coefficients provides a test of the estimation procedure we use, as well as an estimate of the uncertainty in the value of c_0 obtained via Padé methods.

Padé approximation methods are applied to a perturbation series of the form

$$S(x) = 1 + R_1 x + R_2 x^2 + R_3 x^3 + \ldots \tag{9}$$

where R_1 and R_2 are known from a two-loop calculation. An asymptotic error formula[4] for the Padé predictions established in applications to QCD leads to the Padé prediction of R_3.[2]

$$R_3 = \frac{2R_2^3}{R_1\left(R_1^2 + R_2\right)} \tag{10}$$

A complication in the case we are considering is that R_1, R_2 and hence R_3 are implicitly functions of the quantity $w = m_b^2/\mu^2$

$$R_1(w) = a_0 - a_1\log(w) \quad , \quad R_2(w) = b_0 - b_1\log(w) + b_2\log^2(w) \tag{11}$$

The Padé prediction of the coefficients c_i in (6) is obtained from a least squares fit between the w dependence of the Padé prediction $R_3(w)$ and the perturbative form

$$c_0 - c_1\log(w) + c_2\log^2(w) - c_3\log^3(w) \quad . \tag{12}$$

Thus the Padé prediction of the c_i is obtained by minimizing the following expression, in which $R_3(w)$ is estimated by substitution of (11) into (10):

$$\chi^2\left(c_i\right) = \int_0^1 dw\left[R_3(w) - \left(c_0 - c_1\log(w) + c_2\log^2(w) - c_3\log^3(w)\right)\right]^2 \quad . \tag{13}$$

The resulting Padé estimates of the three-loop coefficients c_i are

$$c_0 = 198.4\ ,\ c_1 = 260.6\ ,\ c_2 = 183.9\ ,\ c_3 = 48.64 \quad . \tag{14}$$

These Padé estimates agree with the RG values (8) for $\{c_1, c_2, c_3\}$ to better than 5% accuracy, suggesting a corresponding uncertainty for the Padé-estimated value of c_0.

Similar or better accuracy in the Padé estimates of RG-accessible coefficients has also been obtained in applications to QCD correlation functions and Higgs decay rates.[1,5]

Using the eq. (14) values of c_i, we find the three-loop Padé estimate of the decay rate exhibits reduced renormalization-scale (μ) dependence compared to the two-loop prediction.[2] The significance of the three-loop effects can be assessed by comparing the two-loop decay rate (5) with the three-loop Padé estimate of the decay rate at the renormalization scale $\mu = m_b$

$$\Gamma = K m_b^5 \left(m_b\right) \left[1 + 0.30 + 0.14 + 0.08\right] \quad . \tag{15}$$

However, the choice of renormalization scale $\mu = m_b$ is not necessarily optimal.[3] For an improved prediction we use the minimal sensitivity value of μ where Γ is stable under μ variations. QCD inputs for obtaining this minimal-sensitivity prediction use the four-loop β function[6] and anomalous mass dimension[7] to evolve α and m_b numerically to the scale μ from the values $\alpha\left(M_Z\right)$ and $m_b\left(m_b\right) = 4.17\,\mathrm{GeV}$,[8] with matching conditions through thresholds when necessary.[9] The minimal sensitivity value of the decay rate occurs near the τ mass at $\mu = 1.775\,\mathrm{GeV}$, leading to the Padé determination of the three-loop inclusive semileptonic decay rate:

$$\frac{\Gamma}{K} = \left[5.1213\,\mathrm{GeV}\right]^5 \left[1 - 0.6455 + 0.2477 - 0.0143\right] = 2071\,\mathrm{GeV}^5 \tag{16}$$

The theoretical uncertainties in this Padé determination of the decay rate involve higher-order perturbative effects, uncertainty in the Padé determination of c_0, uncertainty in $\alpha\left(M_Z\right)$ and $m_b\left(m_b\right)$, and nonperturbative contributions, leading to an estimate of the decay rate[2]

$$\frac{\Gamma\left(b \to X_u \ell^- \bar{\nu}_\ell\right)}{K} = \left(2065 \pm 14\%\right)\,\mathrm{GeV}^5 \quad , \quad K = G_F^2 |V_{ub}|^2 / 192\pi^3 \quad , \tag{17}$$

from which $|V_{ub}|$ can be extracted with 7% theoretical uncertainty.

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